

# Black-Scholes Greeks

Harshad Deo

[harshad@simianquant.com](mailto:harshad@simianquant.com)

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This monograph derives the algebraic expressions for the common greeks of the Black-Scholes formula for a vanilla option. Additionally, the formula to invert the pips delta to obtain the strike is also derived. Let

- $S$  be the spot price of the asset
- $K$  be the strike of the vanilla option
- $r_a$  be the terminal drift of the asset
- $r_n$  be the terminal drift of the numeraire
- $\tau$  be the time to expiry of the option
- $\sigma$  be terminal volatility of the asset

The price of the option in units of the numeraire,  $P$ , also called the pips price, is given by:

$$P = \omega \left[ e^{-r_a \tau} S \Phi(\omega d_1) - e^{-r_n \tau} K \Phi(\omega d_2) \right] \quad (1)$$

The price of the option in units of the  $\tau$ -forward value of the numeraire (i.e.  $e^{r_n \tau}$ ),  $P_F$ , also called the forward price, is given by:

$$P_F = \omega \left[ F \Phi(\omega d_1) - K \Phi(\omega d_2) \right] = e^{r_n \tau} P \quad (2)$$

Where

$$F = S e^{(r_n - r_a) \tau} \quad (3)$$

$$d_1 := \frac{\ln(S/K) + (r_n - r_a + \sigma^2/2) \tau}{\sigma \sqrt{\tau}} = \frac{\ln(F/K) + \sigma^2 \tau / 2}{\sigma \sqrt{\tau}} \quad (4)$$

$$d_2 := d_1 - \sigma \sqrt{\tau} \quad (5)$$

$$\omega := \begin{cases} +1, & \text{if call} \\ -1, & \text{if put} \end{cases} \quad (6)$$

# 1 First Order Greeks

Before deriving the expressions for the greeks, it is useful to prove a few lemmas:

**Lemma 1.1.**

$$\frac{\phi(d_2)}{\phi(d_1)} = \frac{S}{K} e^{(r_n - r_a)\tau} \quad (7)$$

*Proof.*

$$\frac{\phi(d_2)}{\phi(d_1)} = \frac{\exp(-d_2^2/2)}{\exp(-d_1^2/2)} \quad (8)$$

$$= \exp\left(-\frac{1}{2}(d_2^2 - d_1^2)\right) \quad (9)$$

$$= \exp\left(-\frac{1}{2}(d_1 + d_2)(d_2 - d_1)\right) \quad (10)$$

$$= \exp\left(-\frac{1}{2}\left(2\frac{\ln(S/K) + (r_n - r_a)\tau}{\sigma\sqrt{\tau}}\right)(-\sigma\sqrt{\tau})\right) \quad (11)$$

$$= \exp\left(\ln(S/K) + (r_n - r_a)\tau\right) \quad (12)$$

$$= \frac{S}{K} e^{(r_n - r_a)\tau} \quad (13)$$

□

**Lemma 1.2.**

$$e^{-r_a\tau} S\phi(d_1)\alpha - e^{-r_n\tau} K\phi(d_2)\beta = e^{-r_a\tau} S\phi(d_1)(\alpha - \beta) \quad (14)$$

*Proof.*

$$e^{-r_a\tau} S\phi(d_1)\alpha - e^{-r_n\tau} K\phi(d_2)\beta \quad (15)$$

$$= e^{-r_a\tau} S\phi(d_1)\left(\alpha - \beta \frac{e^{-r_n\tau} K\phi(d_2)}{e^{-r_a\tau} S\phi(d_1)}\right) \quad (16)$$

$$= e^{-r_a\tau} S\phi(d_1)\left(\alpha - \beta \frac{K}{S e^{(r_n - r_a)\tau}} \frac{\phi(d_2)}{\phi(d_1)}\right) \quad (17)$$

$$= e^{-r_a\tau} S\phi(d_1)\left(\alpha - \beta \frac{K}{S e^{(r_n - r_a)\tau}} \frac{S}{K} e^{(r_n - r_a)\tau}\right), \quad \text{using lemma 1.1} \quad (18)$$

$$= e^{-r_a\tau} S\phi(d_1)(\alpha - \beta) \quad (19)$$

□

**Lemma 1.3.**

$$\frac{\partial d_1}{\partial \tau} = \frac{r_n - r_a + \sigma^2/2}{\sigma\sqrt{\tau}} - \frac{d_1}{2\tau} \quad (20)$$

*Proof.*

$$\frac{\partial d_1}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{\ln(S/K) + (r_n - r_a + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad (21)$$

$$= \frac{r_n - r_a + \sigma^2/2}{\sigma\sqrt{\tau}} - \frac{1}{2\tau} \frac{\ln(S/K) + (r_n - r_a + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad (22)$$

$$= \frac{r_n - r_a + \sigma^2/2}{\sigma\sqrt{\tau}} - \frac{d_1}{2\tau} \quad (23)$$

□

**Lemma 1.4.**

$$\frac{\partial d_1}{\partial \sigma} = \frac{-d_2}{\sigma} \quad (24)$$

*Proof.*

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial}{\partial \sigma} \frac{\ln(S/K) + (r_n - r_a + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad (25)$$

$$= \frac{\sigma\tau}{\sigma\sqrt{\tau}} - \frac{1}{\sigma} \frac{\ln(S/K) + (r_n - r_a + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad (26)$$

$$= \frac{\sigma\sqrt{\tau} - d_1}{\sigma} \quad (27)$$

$$= \frac{-d_2}{\sigma} \quad (28)$$

□

## 1.1 Delta

The delta is defined as the first derivative of the pips price with the asset value.

$$\frac{\partial P}{\partial S} = \omega \frac{\partial}{\partial S} \left[ e^{-r_a\tau} S \Phi(\omega d_1) - e^{-r_n\tau} K \Phi(\omega d_2) \right] \quad (29)$$

$$= \omega \left[ e^{-r_a\tau} \Phi(\omega d_1) + e^{-r_a\tau} S \phi(d_1) \omega \frac{\partial d_1}{\partial S} - e^{-r_n\tau} K \phi(d_2) \omega \frac{\partial d_2}{\partial S} \right] \quad (30)$$

$$= \omega \left[ e^{-r_a\tau} \Phi(\omega d_1) + e^{-r_a\tau} S \phi(d_1) \omega \left( \frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S} \right) \right] \quad \text{using lemma 1.2} \quad (31)$$

$$= \omega \left[ e^{-r_a\tau} \Phi(\omega d_1) + e^{-r_a\tau} S \phi(d_1) \omega \left( \frac{\partial(d_1 - d_2)}{\partial S} \right) \right] \quad (32)$$

$$= \omega \left[ e^{-r_a\tau} \Phi(\omega d_1) + e^{-r_a\tau} S \phi(d_1) \omega \left( \frac{\partial \sigma \sqrt{\tau}}{\partial S} \right) \right] \quad (33)$$

$$= \omega e^{-r_a\tau} \Phi(\omega d_1) \quad (34)$$

## 1.2 Forward Delta

The forward delta is defined as the first derivative of the forward pips price with the forward asset value

$$\frac{\partial P_F}{\partial F} = e^{r_n\tau} \frac{\partial P}{\partial S} \frac{\partial S}{\partial F} \quad (35)$$

$$= e^{r_n\tau} \omega e^{-r_a\tau} \Phi(\omega d_1) e^{(r_a - r_n)\tau} \quad (36)$$

$$= \omega \Phi(\omega d_1) \quad (37)$$

### 1.3 Simple Delta

The simple delta is a simplified form of the forward delta and is useful to parameterize certain volatility models, like local volatility. Let  $\Delta_S$  be the simple delta.

$$\Delta_S := \omega \Phi\left(\omega \frac{\ln(F/K)}{\sigma\sqrt{\tau}}\right) \quad (38)$$

### 1.4 Dual Delta

The dual delta is defined as the first derivative of the pips price with the option strike.

$$\frac{\partial P}{\partial K} = \omega \frac{\partial}{\partial K} \left[ e^{-ra\tau} S \Phi(\omega d_1) - e^{-rn\tau} K \Phi(\omega d_2) \right] \quad (39)$$

$$= \omega \left[ -e^{-rn\tau} \Phi(\omega d_2) + e^{-ra\tau} S \phi(d_1) \omega \frac{\partial d_1}{\partial K} - e^{-ra\tau} K \phi(d_2) \omega \frac{\partial d_2}{\partial K} \right] \quad (40)$$

$$= \omega \left[ -e^{-rn\tau} \Phi(\omega d_2) + e^{-ra\tau} S \phi(d_1) \omega \left( \frac{\partial d_1}{\partial K} - \frac{\partial d_2}{\partial K} \right) \right] \quad \text{using lemma 1.2} \quad (41)$$

$$= \omega \left[ -e^{-rn\tau} \Phi(\omega d_2) + e^{-ra\tau} S \phi(d_1) \omega \left( \frac{\partial(d_1 - d_2)}{\partial K} \right) \right] \quad (42)$$

$$= \omega \left[ -e^{-rn\tau} \Phi(\omega d_2) + e^{-ra\tau} S \phi(d_2) \omega \left( \frac{\partial \sigma \sqrt{\tau}}{\partial K} \right) \right] \quad (43)$$

$$= -\omega e^{-rn\tau} \Phi(\omega d_2) \quad (44)$$

### 1.5 Dual Forward Delta

The dual forward delta is defined as the first derivative of the forward pips price with the option strike

$$\frac{\partial P_F}{\partial K} = e^{rn\tau} \frac{\partial P}{\partial K} \quad (45)$$

$$= e^{rn\tau} (-\omega) e^{-rn\tau} \Phi(\omega d_2) \quad (46)$$

$$= -\omega \Phi(\omega d_2) \quad (47)$$

### 1.6 Vega

The vega is defined as the first derivative of the pips price with terminal volatility.

$$\frac{\partial P}{\partial \sigma} = \omega \frac{\partial}{\partial \sigma} \left[ e^{-ra\tau} S \Phi(\omega d_1) - e^{-rn\tau} K \Phi(\omega d_2) \right] \quad (48)$$

$$= \omega \left[ e^{-ra\tau} S \phi(d_1) \omega \frac{\partial d_1}{\partial \sigma} - e^{-ra\tau} K \phi(d_2) \omega \frac{\partial d_2}{\partial \sigma} \right] \quad (49)$$

$$= e^{-ra\tau} S \phi(d_1) \frac{\partial(d_1 - d_2)}{\partial \sigma} \quad \text{using lemma 1.2} \quad (50)$$

$$= e^{-ra\tau} S \phi(d_1) \frac{\partial \sigma \sqrt{\tau}}{\partial \sigma} \quad (51)$$

$$= e^{-ra\tau} S \sqrt{\tau} \phi(d_1) \quad (52)$$

## 1.7 Forward Vega

The forward vega is defined as the first derivate of the forward pips price with terminal volatility.

$$\frac{\partial P_F}{\partial \sigma} = e^{r_n \tau} \frac{\partial P}{\partial \sigma} \quad (53)$$

$$= e^{r_n \tau} e^{-r_a \tau} S \sqrt{\tau} \phi(d_1) \quad (54)$$

$$= F \sqrt{\tau} \phi(d_1) \quad (55)$$

## 1.8 Rho

The rho is defined as the first derivative of the pips price with the terminal drift of the numeraire.

$$\frac{\partial P}{\partial r_n} = \omega \frac{\partial}{\partial r_n} \left[ e^{-r_a \tau} S \Phi(\omega d_1) - e^{-r_n \tau} K \Phi(\omega d_2) \right] \quad (56)$$

$$= \omega \left[ \tau e^{-r_n \tau} K \Phi(\omega d_2) + e^{-r_a \tau} S \phi(d_1) \omega \frac{\partial d_1}{\partial r_n} - e^{-r_n \tau} K \phi(d_2) \omega \frac{\partial d_2}{\partial r_n} \right] \quad (57)$$

$$= \omega \left[ \tau e^{-r_n \tau} K \Phi(\omega d_2) + e^{-r_a \tau} S \phi(d_1) \omega \frac{\partial (d_1 - d_2)}{\partial r_n} \right], \quad \text{using lemma 1.2} \quad (58)$$

$$= \omega \left[ \tau e^{-r_n \tau} K \Phi(\omega d_2) + e^{-r_a \tau} S \phi(d_1) \omega \frac{\partial \sigma \sqrt{\tau}}{\partial r_n} \right] \quad (59)$$

$$= \omega \tau e^{-r_n \tau} K \Phi(\omega d_2) \quad (60)$$

## 1.9 Phi

The phi is defined as the first derivative of the pips price with the terminal drift of the asset.

$$\frac{\partial P}{\partial r_a} = \omega \frac{\partial}{\partial r_a} \left[ e^{-r_a \tau} S \Phi(\omega d_1) - e^{-r_n \tau} K \Phi(\omega d_2) \right] \quad (61)$$

$$= \omega \left[ -\tau e^{-r_a \tau} S \Phi(\omega d_1) + e^{-r_a \tau} S \phi(d_1) \omega \frac{\partial d_1}{\partial r_a} - e^{-r_n \tau} K \phi(d_2) \omega \frac{\partial d_2}{\partial r_a} \right] \quad (62)$$

$$= \omega \left[ -\tau e^{-r_a \tau} S \Phi(\omega d_1) + e^{-r_a \tau} S \phi(d_1) \omega \frac{\partial (d_1 - d_2)}{\partial r_a} \right], \quad \text{using lemma 1.2} \quad (63)$$

$$= \omega \left[ -\tau e^{-r_a \tau} S \Phi(\omega d_1) + e^{-r_a \tau} S \phi(d_1) \omega \frac{\partial \sigma \sqrt{\tau}}{\partial r_a} \right] \quad (64)$$

$$= -\omega \tau e^{-r_a \tau} S \Phi(\omega d_1) \quad (65)$$

## 1.10 Theta

The theta is defined as the negative of the first derivative of the pips price with time to expiry.

$$-\frac{\partial P}{\partial \tau} = -\omega \frac{\partial}{\partial \tau} \left[ e^{-r_a \tau} S \Phi(\omega d_1) - e^{-r_n \tau} K \Phi(\omega d_2) \right] \quad (66)$$

$$= -\omega \left[ -r_a e^{-r_a \tau} S \Phi(\omega d_1) + r_n e^{-r_n \tau} K \Phi(\omega d_2) + S e^{-r_a \tau} \phi(d_1) \omega \frac{\partial d_1}{\partial \tau} - K e^{-r_n \tau} \phi(d_2) \omega \frac{\partial d_2}{\partial \tau} \right] \quad (67)$$

$$= -\omega \left[ -r_a e^{-r_a \tau} S \Phi(\omega d_1) + r_n e^{-r_n \tau} K \Phi(\omega d_2) + S e^{-r_a \tau} \phi(d_1) \omega \frac{\partial(d_1 - d_2)}{\partial \tau} \right] \quad \text{using lemma 1.2} \quad (68)$$

$$= -\omega \left[ -r_a e^{-r_a \tau} S \Phi(\omega d_1) + r_n e^{-r_n \tau} K \Phi(\omega d_2) + S e^{-r_a \tau} \phi(d_1) \omega \frac{\partial \sigma \sqrt{\tau}}{\partial \tau} \right] \quad (69)$$

$$= \omega r_a e^{-r_a \tau} S \Phi(\omega d_1) - \omega r_n e^{-r_n \tau} K \Phi(\omega d_2) - S e^{-r_a \tau} \phi(d_1) \frac{\sigma}{2\sqrt{\tau}} \quad (70)$$

## 1.11 Forward Theta

The forward theta is defined as the negative of the first derivative of the forward pips price with time. This is also called the driftless theta.

$$-\frac{P_F}{\partial \tau} = -\omega \frac{\partial}{\partial \tau} \left[ F \Phi(\omega d_1) - K \Phi(\omega d_2) \right] \quad (71)$$

$$= -\omega \left[ F \phi(d_1) \omega \frac{\partial d_1}{\partial \tau} - K \phi(d_2) \omega \frac{\partial d_2}{\partial \tau} \right] \quad (72)$$

$$= -e^{r_n \tau} \left[ S e^{-r_a \tau} \frac{\partial d_1}{\partial \tau} - K e^{-r_n \tau} \frac{\partial d_2}{\partial \tau} \right], \quad \text{since } \omega^2 = 1 \quad (73)$$

$$= -e^{r_n \tau} S e^{-r_a \tau} \phi(d_1) \frac{\partial(d_1 - d_2)}{\partial \tau}, \quad \text{using lemma 1.2} \quad (74)$$

$$= -F \phi(d_1) \frac{\partial \sigma \sqrt{\tau}}{\partial \tau} \quad (75)$$

$$= -\frac{F \sigma}{2\sqrt{\tau}} \phi(d_1) \quad (76)$$

## 2 Second Order Greeks

### 2.1 Gamma

The gamma is defined as second derivative of the pips price with the asset value.

$$\frac{\partial^2 P}{\partial S^2} = \frac{\partial}{\partial S} \omega e^{-r_a \tau} \Phi(\omega d_1) \quad (77)$$

$$= e^{-r_a \tau} \phi(d_1) \frac{\partial d_1}{\partial S} \quad (78)$$

$$= \frac{1}{S \sigma \sqrt{\tau}} e^{-r_a \tau} \phi(d_1) \quad (79)$$

## 2.2 Forward Gamma

The gamma is defined as second derivative of the forward pips price with the forward asset value.

$$\frac{\partial^2 P_F}{\partial F^2} = \frac{\partial}{\partial F} \omega \Phi(\omega d_1) \quad (80)$$

$$= \frac{1}{F \sigma \sqrt{\tau}} \phi(d_1) \quad (81)$$

## 2.3 Charm

The charm is the negative of the second derivative of the pips price, once with spot and once with time to expiry. Since the parameters are constant, the order of differentiation does not matter.

$$-\frac{\partial^2 P}{\partial S \partial \tau} = -\frac{\partial}{\partial \tau} \left[ \omega e^{-r_a \tau} \Phi(\omega d_1) \right] \quad (82)$$

$$= \omega r_a e^{-r_a \tau} \Phi(\omega d_1) - e^{-r_a \tau} \phi(d_1) \frac{\partial d_1}{\partial \tau}, \quad \text{since } \omega^2 = 1 \quad (83)$$

$$= \omega r_a e^{-r_a \tau} \Phi(\omega d_1) - e^{-r_a \tau} \phi(d_1) \left( \frac{r_n - r_a + \sigma^2/2}{\sigma \sqrt{\tau}} - \frac{d_1}{2\tau} \right), \quad \text{using lemma 1.3} \quad (84)$$

$$= e^{-r_a \tau} \left[ \omega r_a \Phi(\omega d_1) - \phi(d_1) \left( \frac{r_n - r_a + \sigma^2/2}{\sigma \sqrt{\tau}} - \frac{d_1}{2\tau} \right) \right] \quad (85)$$

## 2.4 Vanna

The vanna is the second derivative of the pips price, once with spot and once with terminal volatility. Since the parameters are constant, the order of differentiation does not matter.

$$\frac{\partial^2 P}{\partial S \partial \sigma} = \frac{\partial}{\partial \sigma} \omega e^{-r_a \tau} \Phi(\omega d_1) \quad (86)$$

$$= e^{-r_a \tau} \phi(d_1) \frac{\partial d_1}{\partial \sigma}, \quad \text{since } \omega^2 = 1 \quad (87)$$

$$= -e^{-r_a \tau} \phi(d_1) \frac{d_2}{\sigma}, \quad \text{using lemma 1.4} \quad (88)$$

## 2.5 Forward Vanna

The forward vanna is the second derivative of the forward pips price, once with the forward and once with the terminal volatility. Since the parameters are constant, the order of differentiation does not matter.

$$\frac{\partial^2 P_F}{\partial F \partial \sigma} = e^{r_n \tau} \frac{\partial^2 P}{\partial S \partial \sigma} \frac{\partial S}{\partial F} \quad (89)$$

$$= e^{r_n \tau} \left( -e^{-r_a \tau} \phi(d_1) \frac{d_2}{\sigma} \right) e^{(r_a - r_n) \tau} \quad (90)$$

$$= -\phi(d_1) \frac{d_2}{\sigma} \quad (91)$$

## 2.6 Volga

The volga (aka vomma) is second derivative of the pips price with terminal volatility.

$$\frac{\partial^2 P}{\partial \sigma^2} = \frac{\partial}{\partial \sigma} e^{-r_a \tau} S \sqrt{\tau} \phi(d_1) \quad (92)$$

$$= e^{-r_a \tau} S \sqrt{\tau} \frac{\partial}{\partial \sigma} \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \quad (93)$$

$$= -e^{-r_a \tau} S \sqrt{\tau} \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} d_1 \frac{\partial d_1}{\partial \sigma} \quad (94)$$

$$= e^{-r_a \tau} S \sqrt{\tau} \phi(d_1) \frac{d_1 d_2}{\sigma}, \quad \text{using lemma 1.4} \quad (95)$$

## 2.7 Forward Volga

The forward volga (aka forward vomma) is the second derivative of the forward pips price with terminal volatility.

$$\frac{\partial^2 P_F}{\partial \sigma^2} = e^{r_n \tau} \frac{\partial^2 P}{\partial \sigma^2} \quad (96)$$

$$= e^{r_n \tau} e^{-r_a \tau} S \sqrt{\tau} \phi(d_1) \frac{d_1 d_2}{\sigma} \quad (97)$$

$$= F \sqrt{\tau} \phi(d_1) \frac{d_1 d_2}{\sigma} \quad (98)$$

## 3 Strike For Delta

### 3.1 Spot Delta

Let  $\Delta$  be the pips spot delta.

$$\Delta := \omega e^{-r_a \tau} \Phi(\omega d_1) \quad (99)$$

$$\therefore d_1 = \omega \Phi^{-1}(\omega e^{r_a \tau} \Delta), \quad \text{since } \omega^2 = 1 \quad (100)$$

$$\therefore \frac{\ln(S/K) + (r_n - r_a + \sigma^2/2)\tau}{\sigma \sqrt{\tau}} = \omega \Phi^{-1}(\omega e^{r_a \tau} \Delta) \quad (101)$$

$$\therefore \ln(K) = \ln(S) + (r_n - r_a + \sigma^2/2)\tau - \omega \sigma \sqrt{\tau} \Phi^{-1}(\omega e^{r_a \tau} \Delta) \quad (102)$$

$$\therefore K = S \exp \left\{ (r_n - r_a + \sigma^2/2)\tau - \omega \sigma \sqrt{\tau} \Phi^{-1}(\omega e^{r_a \tau} \Delta) \right\} \quad (103)$$

### 3.2 Forward Delta

Let  $\Delta_F$  be the pips forward delta.

$$\Delta_F := \omega \Phi(\omega d_1) \quad (104)$$



$$\therefore d_1 = \omega \Phi^{-1}(\omega \Delta_F), \quad \text{since } \omega^2 = 1 \quad (105)$$

$$\therefore \frac{\ln(F/K) + \sigma^2 \tau / 2}{\sigma \sqrt{\tau}} = \omega \Phi^{-1}(\omega \Delta_F) \quad (106)$$

$$\therefore \ln K = \ln F + \frac{\sigma^2 \tau}{2} - \omega \sigma \sqrt{\tau} \Phi^{-1}(\omega \Delta_F) \quad (107)$$

$$\therefore K = F \exp \left\{ \frac{\sigma^2 \tau}{2} - \omega \sigma \sqrt{\tau} \Phi^{-1}(\omega \Delta_F) \right\} \quad (108)$$

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